Int. J. Solids Structures Vol. 29, No. 24, pp. 3145–3152, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

FREE VIBRATIONS OF A FUSED BICONICAL TAPER LIGHTWAVE COUPLER

E. SUHIR
AT&T Bell Laboratories, Murray Hill, NJ 07974, U.S.A.

(*Received* 16 *December* 1991; *in revised form* 6 *June* 1992)

Abstract—A simple formula is obtained for the assessment of the fundamental frequency of lateral vibrations of a fused biconical taper (FBT) lightwave coupler, subjected to initial tensile strain (force). The obtained formula considers the nonprismaticity of the coupler, as well as the nonlinear stress–strain relationship of the material, and can be used to determine the initial strain resulting in a sufficiently high fundamental frequency. This formula can be applied also for the evaluation of the initial tensile force, stress and strain from the measured vibration frequency.

INTRODUCTION

In fused biconical taper (FBT) couplers, the cores of the fibers are positioned very close to each other, so that the two fundamental modes become coupled through their evanescent fields (Miller and Chynoweth, 1979; Sheem and Giallorenzi, 1979; Sheem and Cole, 1979; Bergh *et al.*, 1980; Villarruel and Moeller, 1981; Bures *et al.*, 1983; Bilodeau *et al.*, 1987). In order to bring the cores in close proximity, the cladding in the midportion of the coupler has to be made very thin. At the same time, the coupler must be sufficiently strong, both on a short and long time scale, and must be able to withstand, in addition to thermal and external static loading, dynamic stresses. These can occur during manufacturing, testing, storage, shipment, installation and operation of lightwave couplers, as well as of the devices employing such couplers.

The most common method of ensuring vibration stability of electronic and optical devices is to support and stiffen their structures to an extent when their fundamental natural frequency is sufficiently higher than the highest frequency of the expected excitations. In the case of a FBT coupler, it is the initial strain which "stiffens" the coupler's structure. Such a strain can be caused by the thermal contraction mismatch of the coupler and its base (substrate), or can be even applied deliberately, if there is a need to improve the dynamic stability of the coupler.

In the analysis below we develop a simple formula for the assessment of the fundamental natural frequency of the coupler structure. This formula is intended to be used to establish the initial strain which would result in a sufficiently high vibration frequency. Then a designer should make sure that this strain is acceptable from the standpoint of the long-term reliability ("static fatigue") of the material.

TENSILE FORCE

Let a FBT coupler be subjected to an elongation Δl (Fig. 1). If this elongation is due to the thermal contraction mismatch of the coupler and its base, it can be evaluated using

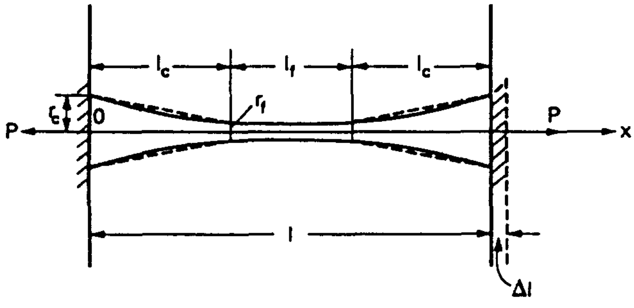

Fig. 1. Fused biconical taper (FBT) lightwave coupler.

an analysis in Appendix A. The equilibrium condition for any portion of the coupler indicates that the resulting axial force P is the same for all the cross-sections of the coupler.

We assume that the actual coupler geometry can be approximated by two circular truncated cones connected by a circular cylindrical midportion, as shown in Fig. 1 in broken lines. Such an approximation is thought to be adequate, as long as the radii r_c and r_f of the larger and the smaller bases of the truncated cones are such that the areas of the corresponding circles are equal to the actual cross-sectional areas. With this assumption, the stress $\sigma(x)$ in any cross-section x of the coupler can be evaluated as

$$\sigma(x) = \frac{P}{\pi r^2(x)} = \sigma_f \frac{r_f^2}{r^2(x)}, \tag{1}$$

where $\sigma_f = P/\pi r_f^2$ is the stress in the fused midportion, and the radius $r(x)$ of the coupler changes over its length as follows:

$$r = \begin{cases} r_c-(r_c-r_f)\dfrac{x}{\ell_c}, & \text{for } 0 \leqslant x \leqslant \ell_c \\ r_f, & \text{for } \ell_c \leqslant x \leqslant \ell_c+\ell_f, \\ r_c-(r_c-r_f)\dfrac{\ell-x}{\ell_c}, & \text{for } \ell_c+\ell_f \leqslant x \leqslant \ell \end{cases} \tag{2}$$

Here ℓ, ℓ_c and ℓ_f are the total length of the coupler, the length of one of its conical parts, and the length of the fused midportion, respectively. The origin of the coordinate x is at the left end of the coupler.

It has been found (Mallinder and Proctor, 1964; Krause *et al.*, 1979; Glasemann *et al.*, 1988) that the elastic behavior of silica fibers subjected to tensile strains not exceeding 5% can be described by the relationship:

$$\sigma = E_0\varepsilon(1+\tfrac{1}{2}\alpha\varepsilon). \tag{3}$$

Here σ is the stress, ε is the strain, E_0 is Young's modulus of the material for very low strains (when the nonlinear behavior of the material need not be accounted for), and α is the parameter of nonlinearity. From (3) we obtain:

$$\varepsilon = \frac{1}{\alpha}\left(\sqrt{1+2\alpha\frac{\sigma}{E_0}}-1\right) = \frac{1}{\alpha}\left(\sqrt{1+\beta^2\frac{r_f^2}{r^2}}-1\right), \tag{4}$$

where

$$\beta^2 = 2\alpha\frac{\sigma_f}{E_0} = 2\alpha\frac{P}{\pi E_0 r_f^2}. \tag{5}$$

Using (4), we present the overall elongation $\Delta\ell$ of the coupler as

$$\Delta\ell = \int_0^\ell \varepsilon(x)\,dx = \frac{1}{\alpha}\left(\int_0^\ell \sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dx-\ell\right). \tag{6}$$

The integral in this equation can be written as follows:

$$\int_0^\ell \sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dx = 2\int_0^{\ell_c} \sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dx+\sqrt{1+\beta^2}\,\ell_f = 2f_c\ell_c+f_f\ell_f, \tag{7}$$

where the factors

$$f_c = \frac{1}{\ell_c}\int_0^{\ell_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dx \tag{8}$$

and

$$f_f = \sqrt{1+\beta^2} \tag{9}$$

consider the effects of the magnitude of the applied force and the material's nonlinearity on the elongations of the conical and the fused portions, respectively. The factor f_c can be evaluated, considering (2), as

$$f_c = \frac{1}{\ell_c}\int_0^{\ell_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dx = \frac{1}{r_c-r_f}\int_{r_f}^{r_c}\sqrt{1+\beta^2\frac{r_f^2}{r^2}}\,dr$$
$$= \frac{1}{1-\rho}\left(\sqrt{1+\beta^2\rho^2}-\rho\sqrt{1+\beta^2}+\beta\rho\ln\frac{\sqrt{1+\beta^2}+\beta}{\sqrt{1+\beta^2\rho^2}+\beta\rho}\right), \tag{10}$$

where $\rho = r_f/r_c$ is the radii ratio. The formulae (6) and (7) result in the following equation which can be used to determine the parameter β, and then the axial force P, for the given overall strain $\Delta\ell/\ell$:

$$\frac{\Delta\ell}{\ell} = \frac{1}{\alpha}\left[\frac{2\ell_c}{\ell}f_c(\beta,\rho)+\frac{\ell_f}{\ell}f_f(\beta)-1\right]. \tag{11}$$

The strain energy responsible for the long-term reliability of the coupler can be evaluated as

$$V_p = \frac{1}{2}\int_0^{\ell}\frac{P^2(x)\,dx}{EA(x)} = \frac{P^2}{2\pi}\int_0^{\ell}\frac{dx}{Er^2}. \tag{12}$$

The expression for Young's modulus E can be found from (3) by differentiation:

$$E = \frac{d\sigma}{d\varepsilon} = E_0(1+\alpha\varepsilon) = E_0\sqrt{1+\beta^2\frac{r_f^2}{r^2}}.$$

Then the formula (12) yields:

$$V_p = \frac{P^2}{2\pi E_0}\int_0^{\ell}\frac{dx}{r\sqrt{\beta^2 r_f^2+r^2}} = 2V_c'\ell_c + V_f'\ell_f, \tag{13}$$

where

$$V_c' = \frac{P^2}{2\pi E_0\ell_c}\int_0^{\ell_c}\frac{dx}{r\sqrt{\beta^2 r_f^2+r^2}} = \frac{P^2}{2\pi E_0}\frac{1}{r_c-r_f}\int_{r_f}^{r_c}\frac{dr}{r\sqrt{\beta^2 r_f^2+r^2}}$$
$$= \frac{P^2}{2\pi E_0(r_c-r_f)\beta r_f^2}\ln\frac{r_c(\sqrt{1+\beta^2}+\beta)}{\beta r_f+\sqrt{\beta^2 r_f^2+r_c^2}} = \frac{\pi E_0 r_f^2}{8\alpha^2}\frac{\rho\beta^3}{1-\rho}\ln\frac{\sqrt{1+\beta^2}+\beta}{\sqrt{1+\beta^2\rho^2}+\beta\rho} \tag{14}$$

and

$$V_f' = \frac{P^2}{2\pi E_0 r^2\sqrt{1+\beta^2}} = \frac{\pi E_0 r_f^2}{8\alpha^2}\frac{\beta^4}{\sqrt{1+\beta^2}} \tag{15}$$

are energies per unit length of the conical and the cylindrical portions of the coupler, respectively. As follows from (14) and (15), the strain energy per unit length of a conical portion is by a factor of

$$\eta = \frac{V_c'}{V_f'} = \frac{\rho\sqrt{1+\beta^2}}{(1-\rho)\beta}\ln\frac{\sqrt{1+\beta^2}+\beta}{\sqrt{1+\beta^2\rho^2}+\beta\rho} \tag{16}$$

smaller than the energy per unit length of the fused midportion.

FUNDAMENTAL FREQUENCY

The total energy of free vibrations of the coupler structure is due to its kinetic energy

$$T = \frac{1}{2}\int_0^{\ell} m(x)\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{17}$$

and the strain energy

$$V = \frac{1}{2}P\int_0^{\ell}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{18}$$

In these formulae, $w = w(x, t)$ are the lateral deflections of the coupler, $m(x) = \pi(\gamma/g)r^2(x)$ is its mass per unit length, γ is the specific weight of the coupler's material, g is the acceleration due to gravity, and $I(x) = (\pi/4)r^4(x)$ is the moment of inertia of the coupler's cross-sectional area. The formula (18) reflects an obvious assumption that the vibration amplitudes are small and therefore the additional strain energy due to axial deformations caused by lateral deflections need not be considered. In addition, it is also assumed that the tensile force is large enough and the flexural rigidity of the coupler is small enough to neglect the strain energy due to bending deformations (see Appendix B).

For the fundamental mode ($i = 1$), the formulae (17) and (18) yield:

$$T = \frac{\gamma}{g}A^2\omega^2\ell^3 C\cos^2\omega t, \quad V = \frac{\pi^2}{4}\frac{P}{\ell}A^2\sin^2\omega t,$$

where the constant C is

$$C = \left(\frac{r_c-r_f}{\ell_c}\right)^2\left\{\frac{1}{6}\left(\frac{\ell_c}{\ell}\right)^3 + \left[\frac{1}{8\pi^3} - \frac{1}{4\pi}\left(\frac{\ell_c}{\ell}\right)^2\right]\sin 2\pi\frac{\ell_c}{\ell} - \frac{1}{4\pi^2}\frac{\ell_c}{\ell}\cos 2\pi\frac{\ell_c}{\ell}\right\}$$
$$-2\frac{r_c}{\ell}\frac{r_c-r_f}{\ell_c}\left[\frac{1}{8\pi^2} + \frac{1}{4}\left(\frac{\ell_c}{\ell}\right)^2 - \frac{1}{4\pi}\frac{\ell_c}{\ell}\sin 2\pi\frac{\ell_c}{\ell} - \frac{1}{8\pi^2}\cos 2\pi\frac{\ell_c}{\ell}\right]$$
$$+\frac{1}{2}\left(\frac{r_c}{\ell}\right)^2\left(\frac{\ell_c}{\ell} - \frac{1}{2\pi}\sin 2\pi\frac{\ell_c}{\ell}\right) + \frac{1}{4}\left(\frac{r_f}{\ell}\right)^2\left[\frac{\ell_f}{\ell} + \frac{1}{2\pi}\sin 2\pi\frac{\ell_c}{\ell} - \frac{1}{2\pi}\sin 2\pi\frac{\ell_c+\ell_f}{\ell}\right].$$

The condition $T_{max} = V_{max}$ results in the following formula for the vibration frequency:

$$\omega = \frac{\pi}{2\ell^2}\sqrt{\frac{g}{\gamma}\frac{P}{C}}. \tag{19}$$

This formula indicates, particularly, that the stress σ_f in the fused portion of the coupler due to the initial strain resulting in the desired (or required) lowest vibration frequency ω, can be evaluated by the formula:

$$\sigma_f = 2\frac{\gamma}{g}\frac{C}{\pi^3}\left(\frac{\omega\ell^2}{r_f}\right)^2. \tag{20}$$

Table 1

β	0	0.02	0.04	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
f_t	1	1.0002	1.0008	1.0018	1.0032	1.0050	1.0198	1.0440	1.0770	1.1180	1.1662	1.2207	1.2806
f_c	1	1.00000	1.00006	1.00013	1.00024	1.00036	1.00296	1.00357	1.00631	1.00979	1.01397	1.01886	1.02429
$\varepsilon_0 = (\sigma_t/E_0)$, %	0	0.0033	0.0133	0.0300	0.0533	0.0833	0.3333	0.7500	1.3333	2.0833	3.0000	4.0833	5.3333
$\varepsilon_t = (f_t - 1/\alpha)$, %	0	0.0033	0.0133	0.0300	0.0533	0.0833	0.3300	0.7333	1.2833	1.9667	2.7700	3.6783	4.6767
$(\Delta\ell/\ell)$, %	0	0.0010 (0.0010)	0.00466 (0.00397)	0.00985 (0.00896)	0.0195 (0.0159)	0.0291 (0.0249)	0.1332 (0.0986)	1.2608 (0.2190)	0.4571 (0.3833)	0.7019 (0.5875)	0.9908 (0.8274)	1.3193 (1.0987)	1.6810 (1.3969)
P, gf	0	0.0773	0.3093	0.6959	1.2372	1.933	7.732	17.40	30.93	48.33	69.59	94.72	123.72

NUMERICAL DATA

(1) The factors f_f and f_c, computed for the radii ratio $\rho = 0.08$, are shown in Table 1. As evident from this table, the f_f values are substantially larger than the f_c values. This is due to the relatively high compliance of the fused midportion. Table 1 also indicates that "actual" (nonlinear) strains ε_f in the fused midportion, calculated, in accordance with the formula (4), for the given force P (or for the given stress σ_f) are appreciably smaller than the "nominal" (linear) strains $\varepsilon_0 = \sigma_f/E_0$.

(2) The forces P and the strains $\Delta\ell/\ell$, shown in Table 1, were obtained for the case $\ell = 38.5$ mm, $\ell_f = 11.5$ mm, $r_f = 0.01$ mm and $r_c = 0.125$ mm, assuming $E_0 = 10.5\times10^6$ psi $= 7384$ kg mm^{-2} $= 72$ GPa, $\alpha = 6$ [see, for instance, Glasemann *et al.* (1988)]. The calculated data show that rather low overall strains $\Delta\ell/\ell$ result in significantly higher strains ε_f in the fused midportion of the coupler. Indeed, let, for instance, the overall displacement of the coupler be $\Delta\ell = 0.0513$ mm, so that $\Delta\ell/\ell = 0.1332\%$. Then, as follows from Table 1 data, the strain in the fused midportion is $\varepsilon_f = 0.33\%$, and the elongation of this portion is $\Delta\ell_f = \varepsilon_f\ell_f = 0.038$ mm. Thus, the elongation of the conical parts, whose total length is $2\ell_c = 27.0$ mm, i.e. by a factor of 2.35 greater than the length of the fused midportion, is $\Delta\ell_c = 0.0513-0.0380 = 0.0133$ mm, which is only about 25% of the overall elongation. Clearly, the stresses in the fused portion can be easily determined from the calculated σ_f/E_0 ratios.

(3) The factor η calculated by the formula (16) for the radii ratio $\rho = (r_f/r_c) = 0.08$ and the nonlinearity parameter $\beta = 0.2$ is only $\eta = 0.081$. Then the total strain energy of the conical parts is only 16% of the entire strain energy, despite the fact that these portions account for about 70% of the coupler's length, and for more than 99% of its volume.

(4) Let us assume, for instance, that the "equivalent" radius of the coupler is $r_0 = (r_c+r_f)/2 = 0.0675$ mm. Then, assuming $E = E_0$ and $i = 1$, and using formula (B6) of Appendix B we obtain $P_* = 0.00387$ gf. Hence, as one can see from Table 1 data, the coupler structure can be indeed considered simply supported at the ends even for very low values of the tensile force P.

(5) Let the highest expected excitation frequency be, say, 2000 Hz. With the factor of safety equal to two, the required fundamental frequency is 4000 Hz. For the coupler in question we obtain: $C = 16.866\times10^{-8}$, $\sigma_f = 67.8$ kg mm^{-2}, $\varepsilon_f = 0.894\%$. Thus, with the chosen factor of safety, the coupler's material should be able to withstand long-term strains of about 0.9%.

CONCLUSION

The developed formula for the fundamental vibration frequency of FBT couplers can be helpful in the structural analysis and mechanical ("physical") design of such couplers.

Acknowledgements—The author acknowledges and thanks R. D. Tuminaro, C. R. Kurkjian, J. T. Krause, L. L. Blyler, Jr and L. T. Manzione for useful discussions and valuable comments.

REFERENCES

Bergh, R. A., Kotler, G. and Shaw, H. J. (1980). Single-mode fibre optic directional coupler. *Electronics Letters* **16**(7), 259–260.

Bilodeau, F., Hill, K. O., Johnson, D. C. and Faucher, S. (1987). Compact, low-loss, fused biconical taper couplers: overcoupled operation and antisymmetric supermode cutoff. *Optics Letters* **12**(8), 634–636.

Bures, J., Lacroix, S. and Lapierre, J. (1983). Analyse d'un coupleur bidirectionrel à fibres optiques monomodes fusionnées. *Appl. Optics* **22**(12), 1918–1922 (in French).

Glasemann, G. S., Gulati, S. T. and Helfinstine, J. D. (1988). Effect of strain and surface composition on Young's modulus of optical fibres. *11th Opt. Fiber Comm. Conf., Techn. Digest*, TUG-5, 26.

Krause, J. T., Testardi, L. R. and Thurston, R. N. (1979). Deviations from linearity in the dependence of elongation upon force for fibers of simple glass formers and of glass optical lightguides. *Phys. Chem. Glasses* **20**(6), 135–139.

Mallinder, F. P. and Proctor, B. A. (1964). Elastic constants of fused silica as a function of large tensile strain. *Phys. Chem. Glasses* **5**(4), 91–103.

Miller, S. E. and Chynoweth, A. G. (1979). *Optical Fiber Telecommunications*. Academic Press, New York.

Sheem, S. K. and Cole, J. H. (1979). Acoustic sensitivity of single-mode optical power dividers. *Optics Letters* **4**(10), 322–324.

Sheem, S. K. and Giallorenzi, T. G. (1979). Single-mode fiber-optical power divider: encapsulated etching technique. *Optics Letters* 4(1), 29–31.

Timoshenko, S. P. and Young, D. H. (1955). *Vibration Problems in Engineering* (3rd Edn). Van Nostrand, Princeton, NJ.

Villarruel, C. A. and Moeller, R. P. (1981). Fused single-mode fibre access couplers. *Electronics Letters* 17(6), 243–244.

APPENDIX A. INITIAL STRAIN CAUSED BY THE THERMAL CONTRACTION MISMATCH OF THE COUPLER AND ITS BASE

Let a coupler manufactured with an initial curvature

$$w(x) = f_0 \sin\frac{\pi x}{\ell}$$

at the elevated temperature be cooled down to the room or testing temperature. In this formula $w(x)$ is the deflection function, ℓ is the length of the coupler's base, and f_0 is the maximum initial deflection of the coupler. The length of the curved coupler can be approximately evaluated as

$$s = \int_0^{\ell} \sqrt{1+[w'(x)]^2}\,dx \cong \int_0^{\ell}\left(1+\frac{1}{2}[w'(x)]^2\right)dx = \ell(1+c^2), \tag{A1}$$

where $c = \pi f_0/2\ell$. Solving (A1) for f_0, we obtain:

$$f_0 = \frac{2}{\pi}\sqrt{s\ell-\ell^2}.$$

The total change in the initial deflection f_0 caused by the change in the length ℓ of the coupler's base and in the length s of the coupler itself can be evaluated as the complete differential

$$df_0 = \frac{\partial f_0}{\partial \ell}d\ell + \frac{\partial f_0}{\partial s}ds = \frac{2\ell}{\pi^2 f}[ds-(1-c^2)\,d\ell]. \tag{A2}$$

The change Δs in the length of the coupler can be found as

$$\Delta s = s-s_0 = s_0(1-\alpha_1\Delta t)-s_0 = -s_0\alpha_1\Delta t, \tag{A3}$$

or, considering (A1),

$$\Delta s = -\ell_0(1+c^2)\alpha_1\Delta t.$$

Here α_1 is the coefficient of thermal expansion (contraction) of the coupler's material, ℓ_0 is the initial length of the base, and Δt is the change in temperature. The change $\Delta\ell$ in the length of the base is

$$\Delta\ell = \ell-\ell_0 = \ell_0(1-\alpha_2\Delta t)-\ell_0 = -\ell_0\alpha_2\Delta t, \tag{A4}$$

where α_2 is the coefficient of thermal expansion of the base material. Since the coupler is made of doped silica, and its base is made of regular silica, α_1 is, as a rule, larger than α_2.

Considering small deflections, we omit in the above formulae the c^2 value, which is small compared to unity, and replace the differentials in (A2) with finite differences. Then eqns (A2), (A3) and (A4) yield:

$$\Delta f = -\frac{\Delta\alpha\Delta t}{\pi c}\ell_0 = -\frac{2\ell_0^2}{\pi^2 f_0}\Delta\alpha\Delta t, \tag{A5}$$

where $\Delta\alpha = \alpha_1-\alpha_2$. Clearly, formula (A5) can be applied as long as the initial deflection f_0 does not exceed its reduction Δf_0. Putting $\Delta f_0 = -f_0$, we find that formula (A5) can be used if the deflection f_0 is larger than

$$f_* = \frac{\ell_0}{\pi}\sqrt{2\Delta\alpha\Delta t}, \tag{A6}$$

otherwise the drop in temperature will result in a tensile strain in the coupler structure. This strain can be determined as a difference between the total thermal contraction mismatch strain $\Delta\alpha\Delta t$ and its portion

$$\varepsilon_* = \frac{\pi^2}{2}\left(\frac{f_0}{\ell_0}\right)^2 \tag{A7}$$

required to bring the initial deflection to zero. Formula (A7) can be obtained from (A6) by solving this equation for $\varepsilon_* = \Delta\alpha\Delta t$.

Thus, the initial tensile strain in the coupler due to its thermal contraction mismatch with the base, can be evaluated by the formula

$$\frac{\Delta\ell}{\ell_0} = \Delta\alpha\Delta t - \frac{\pi^2}{2}\left(\frac{f_0}{\ell}\right)^2. \tag{A8}$$

APPENDIX B. EFFECT OF THE BOUNDARY CONDITIONS AND THE TENSILE FORCE ON THE FUNDAMENTAL FREQUENCY

(1) Consider, for the sake of simplicity, a uniform beam clamped at its ends and subjected to a tensile force P. Using the equation of motion in the form

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0$$

[see, for instance, Timoshenko and Young (1955)], seeking its solution in the form of an expansion

$$w(x,t) = \sum_{i=1}^{\infty} X_i(x)\sin\omega_i t, \tag{B1}$$

and using the boundary conditions

$$X_i(0) = X_i(\ell), \quad X_i'(0) = X_i'(\ell) = 0$$

for the vibration mode function $X_i(x)$, we obtain the following equation for the frequency ω_i of the ith mode of vibrations:

$$2\gamma_i\delta_i(\cosh u_i\cos v_i - 1) + (\delta_i^2 - \gamma_i^2)\sinh u_i \sin v_i = 0. \tag{B2}$$

Here, $u_i = \gamma_i\ell$, $v_i = \delta_i\ell$, and the parameters γ_i and δ_i are related to the frequency ω_i as follows:

$$\left.\begin{aligned}\gamma_i &= \sqrt{\frac{P}{2EI}\left(\sqrt{1+\frac{4EI}{P^2}m\omega_i^2}-1\right)}\\ \delta_i &= \sqrt{\frac{P}{2EI}\left(\sqrt{1+\frac{4EI}{P^2}m\omega_i^2}+1\right)}\end{aligned}\right\}. \tag{B3}$$

These formulae indicate that if the force P significantly exceeds the value

$$P_* = 2\omega_i\sqrt{EIm}, \tag{B4}$$

then $\gamma_i = 0$, $\delta_i = \sqrt{P/EI}$, and eqn (B2) reduces to a frequency equation $\sin v_i = 0$ for a simply-supported bar. In such a case

$$X_i(x) = \sin\frac{i\pi x}{\ell}, \quad \omega_i = \frac{i\pi}{\ell}\sqrt{\frac{P}{m}+\left(\frac{i\pi}{\ell}\right)^2\frac{EI}{m}}, \tag{B5}$$

and formula (B4) yields:

$$P_* = 2(1+\sqrt{2})\left(\frac{i\pi}{\ell}\right)^2 EI = i^2\frac{1+\sqrt{2}}{2}\pi^3 E\frac{r_0^4}{\ell^2}. \tag{B6}$$

(2) Let us now assess the second (bending) term in the equation

$$V = \frac{1}{2}P\int_0^\ell\left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^\ell EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{B7}$$

for the strain energy due to both tension and bending of a beam [see, for instance, Timoshenko and Young (1955)]. Assuming again that the flexural rigidity EI is constant and is equal to $(\pi/4)E_0r_0^4$, seeking the deflection function $w(x,t)$ in the form (B1) and using the first formula in (B5) for the vibration mode, we conclude that the bending term contributes very little to the total strain energy, if the tensile force P is significantly larger than the value

$$P_b = i^2\frac{\pi^3}{2}E\frac{r_0^4}{\ell^2}. \tag{B8}$$

Comparing this formula with (B6), we also conclude that if the condition $P \gg P_*$ is fulfilled, the condition $P \gg P_b$ is fulfilled as well, i.e. *if the tensile force P is large enough so that the coupler can be considered simply supported* at its ends, it is also large enough so that the effect of bending would not have to be accounted for. Thus, for a sufficiently large tensile force ($P \gg P_*$), the strain energy of the coupler can be assessed by a simple formula

$$V = \frac{1}{2}P\int_0^\ell\left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{B9}$$